Admissible Functional Forms in Monetary Economics

A B S T R A C T

This paper introduces the micro foundation concept of 'admissible functional form' for general equilibrium monetary models. To be considered admissible a functional form of a monetary model should possess three properties relative to the unit-free measures derived from the model: 1) these measures should not depend upon the monetary unit of account, 2) in a risk-free steady-state equilibrium the derived unit-free variables should not depend upon the frequency of analysis, and 3) the model's predictions for a unit-free measure should be within the range of empirically observed values for that measure. The unit-free measures considered in this study are the annual velocity of money, the interest and income elasticities of money demand, and the welfare cost of inflation.

The concept of admissibility places restrictions upon the structure of a model that have strong theoretical implications. The contribution of this paper is the derivation of these restrictions for simple versions of money in the utility function and cash-in-advance models. For instance 1) a utility function with money as an argument must be of the CES form, 2) a cash-in-advance model must include a transactions production function that is homogenous of degree zero in money denominated variables, and 3) derived money demand function must be of the log-log form. An important implication of admissibility is that there is a one-to-one equivalence between the predictions of an admissible cash-in-advance and money-in-the-utility function model.

This paper also introduces a general method for building the frequency of analysis into a model.

Keywords: admissible functional form, money-in-the-utility function, cash-in-advance, micro foundation, frequency of analysis

13 September 2007

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1. Introduction

This paper introduces the micro foundation concept of 'admissible functional form' for general equilibrium monetary models. The first requirement for a functional form to be admissible is that its unit-free measures should be neutral with respect to the monetary unit of account. The concept of demand elasticity was introduced by Alfred Marshall (1885). Elasticity allows us to understand the *ceteris paribus* relationship between a price or income and the quantity demanded of a good and to compare this relationship between goods that are measured in different units and that are sold in different currencies. The fact that the elasticity of demand is a unit-free measure places restrictions upon the demand function. For instance, in a general equilibrium macroeconomic model the choice of base year, or whether income is reported in millions of dollars verses billions of dollars, should not affect the derived income or interest elasticities. To satisfy this requirement, a demand function derived within the model should be homogenous of degree one with respect to money-denominated variables; this occurs if, and only if, the income and interest elasticities of money demand are homogenous of degree zero in money denominated variables. This zero-homogeneity requirement should carry over to any unit-free measure such as the annual income velocity of money or a unit-free measure of the welfare cost of inflation. Such a requirement is clearly needed if we use the model for quantitative analysis; beyond this, the requirement insures the internal consistency of the model's results.

The second requirement for a functional form of a model to be admissible is that the model's unit-free measures should be invariant with respect to the frequency of analysis in risk-free steady-state equilibrium. For instance, the calibration of a model using quarterly data and parameter values should yield the same value for unit-free measures of the model as an annual study of the same model. Barring this, the model's predictions are somewhat arbitrary and the model is internally inconsistent. As a result of this requirement I introduce a general method for building the notion of frequency of analysis into the functional form of the model.

The third requirement is that the model's predictions for the observable unit-free measures lie within the range of values that are empirically observed. This requirement is necessary to exclude functional forms that satisfy the first two requirements, but that are clearly not consistent with empirical evidence. To understand the necessity of the third requirement I will demonstrate that the commonly used functional form for the money-in-the-utility function model, the Cobb-Douglas utility function, implies that the interest elasticity of money demand is -1, a value for this parameter which lies outside of the generally accepted range for this measure of -0.05 to -0.50. Mis-specifying the interest elasticity of money demand affects all of the model's monetary predictions.

The question of how to add money to macroeconomic analysis within the Ramsey-Solow neoclassical growth model framework has long intrigued and divided macroeconomists. Two distinct approaches have gained prominence: money-in-the-utility-function (MIUF) and cash-in-advance (CIA). The MIUF model was introduced by Sidrauski (1967). The MIUF model places money directly into the utility function to motivate a demand for money under the principle that the services of money increase the consumer's happiness and that the provision of these services is increasing, at least non-decreasing, in the stock of real money balances. Since the MIUF model is general and does not specify *how* money adds to utility, the services of money in the MIUF model has been viewed as a combination of transactions, precautionary, and portfolio balancing services.

The CIA model motivates a demand for money by positing that some, or all, purchases must be made with cash that was previously obtained. This structure captures the role of money in keeping agents within their budget constraint; money provides a low-cost method of exchange and accounting (Clower, 1967 and Lucas, 1980). While the most rudimentary CIA model predicts a unitary velocity of money, the introduction of risk by Lucas (1980) and Svensson (1985), and credit goods by Lucas and Stokey (1987) allows the predicted velocity to take on values of less than one and more than one, respectively. For these models, inflation causes an efficiency loss by driving a distorting tax wedge between goods consumed in different periods Lucas (1980) and Svensson (1985) or between cash and credit goods in the Lucas and Stokey (1987).

This paper asks 'what properties of functional forms are required for basic versions of a MIUF and a CIA model to be admissible?' Whether a functional form is admissible depends upon the function itself and upon the model in which it is situated. I will show that the utility function for the basic MIUF model *must* be of the CES form and I will show that that a CIA model must include a 'transactions production function' that is homogenous of degree zero in order to be admissible. These are fundamental contributions to the MIUF and CIA literatures. Imposing admissibility has strong theoretical implications for the models' results. For instance, there is a one-to-one equivalence between the predictions of admissible MIUF and CIA models. Secondly, a money demand function must be of the log-log form in order to be consistent with the micro foundation requirements of admissibility. Thirdly, the derived money demand function must display unitary income elasticity. Another contribution of this paper is the introduction of the concept of building the frequency of analysis into a model.

The rest of the paper is organized as follows: section 2 offers examples of admissible functional utility function forms for a MIUF model and functional form for a 'transactions production function' for a CIA model, presents some theorems about the requirements of admissibility for these models, and examines the implications of admissibility. Section 3 concludes.

2. Admissible Forms

Before introducing the admissible functional forms for MIUF and CIA models, the first consideration to be addressed is the concept that a unit-free measure should not depend upon the frequency of analysis, at least in the setting of a risk-free and stationary equilibrium. We expect a stock variable, like money, to be constant with respect to the frequency of analysis, while a flow variable varies in a natural way with the frequency; the quarterly income is $\frac{1}{4}$ of the annual and the monthly income is $\frac{1}{12}$ of the annual. This facet must be built into the model to guarantee that the model's results are independent of the frequency of analysis.

2. 1. A MIUF Model (Model *a*) [1]

Each of the two models analyzed in this study are a representative agent Lucas-style tree economy (Lucas, 1980) with output, *y*, arriving exogenously. Money is supplied according to the process $\overline{M}' = \omega \cdot \overline{M}$ [2] where the monetary innovations are given to consumers in the form of a flat rate transfer payment of $(\omega - 1) \cdot \overline{M}$. The first model, Model *a*, will be explained in detail, while I will only fully explain the innovations particular to the second model, Model *b*.

The representative consumer is subject to the following nominal budget constraint:

$$Pc + M' \leq Py + M + (\omega - 1)\overline{M} \quad . \tag{1a}$$

The right hand side of equation 1a represents the wealth available to the consumer in the period. This consists of, from left to right, the income from the sale of fruit from the tree, money carried into the period and the flat rate transfer payment, or tax, consisting of innovations to the money supply. Wealth is divided between, on the left hand side of equation 1a, consumption and the acquisition of money to carry into the future.

[1] I use a number and letter to identify equations, the letter suffix designates the model.
[2] Time subscripts are suppressed for clarity. Following Svensson (1985) variables dated at time $t + 1$ are denoted with a prime. The bar superscript above *M* signifies money supply as opposed to money demand.

Here, capital letters refer to nominal variables, small letters refer to real variables, and Greek letters represent parameters. To make the problem stationary I divide all terms by this period's money supply, $\overline{M}$, and I define the following: $p = \frac{P}{\overline{M}}$ and $m = \frac{M}{\overline{M}}$; notice that $\frac{M'}{\overline{M}} = \omega m'$. Subsequent analysis of both models will be presented in these real terms. This transformation yields the following stationary real valued budget constraint:

$$pc + \omega m' = py + m + \omega - 1 \ . \tag{2a}$$

In market-clearing equilibrium the ratio of money demand to money supply is one and all output is consumed:

$$\begin{aligned} m' &= m = 1 \\ c &= y \ . \end{aligned} \tag{3a}$$

The representative consumer's problem is expressed as a value function problem:

$$v(S) = \max_{c,\, m'} \left\{ U\left(c, \frac{m}{p}\right) + \beta \cdot Ev(S') \right\} \tag{4a}$$

where $S = \{\omega, y\}$ represents the state of the economy and β is the rate at which the consumer discounts future consumption relative to current consumption. The parameters ω and β are appropriate to the frequency of the data, so for instance if the consumer's rate of time preference is 4% per year then $\beta = 0.96$ in an annual study and $\beta = 0.99$ in a quarterly study.

The consumer seeks to maximize the value function subject to the budget constraint. The first-order conditions for the consumer's problem are given below, where λ is the Lagrange multiplier for the budget constraint:

$$v_c(S) = U_c - p\lambda = 0 \tag{5a}$$

$$v_{m'}(S) = -\omega\lambda + \beta E\, v_{m'}(S') = 0 \tag{6a}$$

here E is the expectations operator and the $v_{m'}(S')$ term is found using the envelope theorem:

$$E\, v_{m'}(S') = E[U_{m'} + \lambda'] \ . \tag{7a}$$

To simplify the analysis and reduce notation, throughout the paper I will only consider risk-free steady-state equilibria, those in which the *level* of output and the *rate* of money growth are viewed as constant, $y' = y$ and $\omega' = \omega$.[3] As a result of this assumption we expect to obtain steady values for the endogenous prices and Lagrange multiplier, so $p' = p$ and

[3] Arnwine (2007) considers the case with output growth, allowing output growth adds to the required notation without changing this paper's findings.

$\lambda' = \lambda$. Restricting the analysis to steady-state equilibria is motivated in part by the fact that the second requirement of admissibility is evaluated in such an equilibrium.

The first-order conditions are combined with the market-clearing restrictions to obtain the following Euler equations:

$$\lambda = \frac{U_c}{p} \tag{8a}$$

$$\omega\lambda = \beta\left[U_m(c', m') + \lambda'\right] \ . \tag{9a}$$

The final solutions will be cleaner if I define the shadow nominal interest rate as the rate of return on a hypothetical nominal bond:

$$i = \frac{\omega}{\beta} - 1 \ . \tag{10a}$$

This is Fisher's relation (Fisher, 1896) and is standard in the literature; to defer spending one dollar for one period the consumer must be given a gross real return of β^{-1} which requires a gross nominal return of $\omega\beta^{-1}$, where ω is simultaneously (one plus) the money growth rate and (one plus) the inflation rate. The rate i is the rate per period of observation. This relation/definition will be utilized throughout the paper. After imposing the stationarity conditions equations 9a and 10a yield:[4]

$$\lambda = \frac{U_m}{i} \tag{11a}$$

which states that the consumer collects real money balances until the marginal utility of money equals $i \cdot \lambda$, which can be interpreted as the rental rate of a marginal dollar of income, measured in utility terms. Combining equations 8a and 11a we obtain:

$$p = i \cdot \frac{U_c}{U_m} \ . \tag{12a}$$

Here p is the deflated money price of output. This price level depends upon the ratio of the marginal utilities of consumption to real money balances and the nominal interest rate. Below, this equilibrium price is used to define the equilibrium income velocity of money and level of real money balances.

Since the velocity of money must be homogenous of degree zero and the demand for real balances must be linearly homogenous, we will see that p^{-1} must also be linearly homogenous. Therefore the utility function must be of the CES form, or an affine

[4] Stationary implies that the marginal utilities are constant over time, so the time notation has been dropped from equation 11a and subsequent equations in this section.

transformation of the CES form, to insure that the ratio of marginal utilities is constant with respect to changes in output. Theorem 1, presented below, formalizes this proposition. This is a contribution to the MIUF model literature.

The proofs of Theorems 1 and 2 utilize the derived unit-free measures of the model, so I first proceed by defining the MIUF utility function and derive the model's unit-free measures for this functional form. Theorem 1 demonstrates that this utility function is a very general representation of the admissible utility functions for the model outlined in equations 2a-4a:

$$U\left(c,\frac{m}{p}\right)=\frac{1}{\phi(1-\alpha)}\cdot\left[\left(\phi c\right)^{\frac{\gamma-1}{\gamma}}+\theta\cdot\left(\frac{m}{p}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}(1-\alpha)}. \tag{13a}$$

The parameter ϕ is not a free parameter, rather ϕ represents the number of observation periods per year, so that $\phi = 1$ for a model with annual frequency, equals $\phi = 4$ for a quarterly model and $\phi = 12$ for a monthly model. The flow variable of the objective function, c, is pre-multiplied by this parameter; this is required for the model to satisfy the second requirement of admissibility that requires unit-free measures to be invariant to the frequency of analysis. This transformation has the effect of annualizing data of other frequencies, for instance quarterly income should be one-quarter of annual income, but this is pre-multiplied by 4 in the utility function.[5] The period utility is divided by ϕ because we expect the utility accruing in a quarter to be one-fourth of the utility accruing in a year. A key element of monetary models is the interaction of a stock variable, money, and flow variables, such as income. Including the ϕ parameter and recognizing of the concept of 'frequency of analysis' are required in general for the model's predictions of unit-free variables to be invariant to the frequency of analysis.

The parameter γ is the elasticity of substitution between the utility function inputs of consumption and real money balances. We will see that the interest elasticity of money demand is $-\gamma$, a fact which can be used to calibrate the model. The inclusion of this parameter in the utility function implies that inflation is important to welfare because of its effect on money demand. The free parameter θ is calibrated to fit the velocity of money.

Now we can explicitly solve the pricing equation, equation 12a:

[5] The conversion of the interest rate does not involve compounding; a model's notation becomes very messy when combining a mix of compounded and non-compounded variables; however, we will see that this is not an issue in the end because admissibility requires us to use annualized data in the analysis. The annualized data series should be constructed according to the principle that is proper to the data type, whether it be a rate of return, a flow, or a stock.

$$p = \frac{\tilde{\theta}}{\phi y} \cdot (\phi i)^{\gamma} \tag{14a}$$

where $\tilde{\theta} \equiv \theta^{-\gamma}$. The annualized velocity of money is:

$$V \equiv \frac{P \phi y}{\overline{M}} = p \phi y = \tilde{\theta}(\phi i)^{\gamma} \tag{15a}$$

and the level of real balances is:

$$L \equiv \frac{M}{P} = \frac{1}{p} = \tilde{\theta}^{-1}(\phi i)^{-\gamma} \cdot \phi y \quad . \tag{16a}$$

The derived income velocity of money is homogenous of degree zero in money-valued variables and the derived money demand function is linearly homogenous in money-valued variables. Both measures are unaffected by a change in the frequency of the data; the terms ϕi and ϕy can be viewed as annualized observations.

The output elasticity and interest elasticity of money demand are derived from the money demand function, equation 16a. The income elasticity of the demand for real balances is 1. One theoretical implication of the notion of admissibility is that we *expect* to observe a unitary income elasticity of money demand in this MIUF model. On one hand, since the income velocity of money is homogenous of degree zero in the money-denominated variable, y, a *ceteris paribus* increase in equilibrium income does not change the velocity of money. Additionally, since the nominal money supply is viewed as exogenous, an increase in income must proportionally decrease the pricing function, p, to maintain the exchange equation identity and, thus, proportionally increase the level of real balances. We observe these relations in equations 14a and 16a. If we wish to produce a model in which the income elasticity of money demand is not unitary then we need to alter the underlying MIUF model, not merely the functional form for utility. Perhaps we could make the nominal money supply endogenous as in Freeman and Kydland (2000) or add more money-denominated variables so that money velocity changes with income, but is still homogenous of degree zero in money-denominated variables. This is the first of a number of theoretical implications resulting from the imposition of admissibility on model a.

The interest elasticity of money demand is found by taking the partial derivative of money demand, equation 16a, with respect to the interest rate and multiplying the result by the ratio of interest to real money balances:

$$\varepsilon_{Li} = \frac{\partial L}{\partial i} \cdot \frac{i}{L} = -\gamma \quad . \tag{17a}$$

The interest elasticity of money demand is determined by the consumer's elasticity of substitution between consumption and real money balances.

The utility accruing in a period under the optimal consumption rules is given by the derived indirect utility function, U^*, which is found by substituting the optimal rules into the utility function, U. Combining equations 13a, 3a, and 16a we obtain:

$$U^*(i,y) = \frac{1}{1-\alpha} \cdot \left[1 + \tilde{\theta}^{-1} \cdot (\phi i)^{\gamma-1}\right]^{\frac{\gamma}{\gamma-1}(1-\alpha)} \cdot (\phi y)^{1-\alpha} \quad . \tag{18a}$$

The optimal inflation rate for this model is the one that yields a nominal interest rate of zero. This is consistent with the analysis of Bailey (1956) and Friedman (1969). The intuition is that the cost of supplying money is practically zero so it should be supplied to the extent that the opportunity cost of holding money for the consumer is zero. The nominal interest rate is the opportunity cost of holding money, so $i = 0$ under the optimal policy. Because of our Fisher relation, equation 10a, we have $\omega = \beta$ when following the optimal money supply rule. Money should be taken out of circulation at the rate of the consumer's time preference so that it gains value at that same rate. The rate of optimal money appreciation would change if the model included income growth (Arnwine and Yigit, 2007), if money earns interest (Bewley, 1983), or if there were distortionary taxes in the model, (Phelps, 1973 and Cooley and Hansen, 1989, 1991). These cases are not considered in this paper.

Two measures of the welfare cost of inflation are contemplated: the first is defined relative to an 'inflation compensating level of income', $\hat{y}$:

$$\Pi(i) \equiv \frac{\hat{y}}{y} - 1 = \left[1 + \tilde{\theta}^{-1} \cdot (\phi i)^{\gamma-1}\right]^{\frac{\gamma}{1-\gamma}} - 1 \tag{19a}$$

where $\hat{y}$ is defined to be the level such that $U^*(i,\hat{y}) = U^*(0,y)$. The welfare cost of inflation is the proportion increase in output that would be required to compensate the consumer for living with interest rate i, rather than the optimal rate of $i = 0$. This 'welfare cost of inflation' depends upon the parameters governing the elasticity of money and the velocity of money, so it is evident that inflation is important to welfare to the extent that it causes the consumer to alter his or her level of real money balance.

The second measure of the welfare cost of inflation is based upon Bailey (1956). Bailey's measure consists of the area of consumer surplus under the demand curve, equation 16a, that is lost when $i > 0$. I divide the lost consumer surplus by the income level to obtain a unit-free measure. The integration and normalization are done in annualized terms in equation

20a; changing the frequency of analysis does not change the result, as long as we are consistent:

$$\Omega(i) \equiv \frac{-1}{\phi y} \cdot \int_0^{\phi i} L(y,i)\, di = \frac{1}{\gamma \tilde{\theta}} \cdot (\phi i)^{1-\gamma} \quad . \tag{20a}$$

The welfare cost of a given level of inflation is decreasing in the absolute value of the interest elasticity of money demand, γ, and is decreasing in $\tilde{\theta}$ which is calibrated to match the velocity of money; inflation matters less if the consumer does not change his behavior as a result.

The admissible functional form for the simple MIUF model leads to parsimonious functional forms for the model's unit-free measures: the income velocity of money, the income and interest elasticities of money, and the welfare cost of inflation. These four unit-free measures are invariant to the frequency of data used in the analysis because flow variables in these expressions are pre-multiplied by the ϕ term. These unit-free measures are all homogenous of degree zero in money-denominated variables, so they do not vary with the monetary measure.

The utility function, equation 13a, includes two free parameters, γ and θ, in addition to the utility function and model parameters, α and β, that are standard in time-dynamic models and the parameter for money growth, ω, which is standard for monetary models. The inclusion of the parameters γ and θ means that the model can exactly match average data values for two observed measures, the interest elasticity of money demand and the income velocity of money.

Theorem 1 establishes that a utility function in consumption and real balances for the model outlined in equations 2a - 4a has unit-free measures that are invariant to the monetary unit of account if and only if the utility function is the CES form or an affine transformation of the CES form. This, for instance, excludes the commonly used additively separable utility functions from the set of admissible MIUF functional forms. Some open questions in monetary economics may be resolved by this restriction. For instance Obstfeld and Rogoff (1986) show that purely speculative hyperinflation is possible unless the utility function is restricted to have negative infinite utility when the money stock is zero. The possibility of a purely speculative hyperinflation is eliminated when we restrict the utility function to the

CES form. Limiting the functional form of the MIUF structure to the CES form also eliminates the possibility of non-monetary equilibria, and multiple equilibria, Brock (1974).[6]

<Theorem 1 Here>

Step ii (b) of Theorem 1 demonstrates that the elasticity of substitution between real balances and consumption in the consumer's utility function is the negative of the derived interest elasticity of money demand. Money 'matters' in MIUF models because, and to the extent that, it is a substitute for consumption. Therefore, micro foundation restrictions on the interest elasticity of money demand restrict the functional form of the utility function.

Analysis of the proof of Theorem 1 indicates that we could generalize the original MIUF model analyzed in this section by taking the degree of substitution between c and L in the utility function as a function of the interest rate or of any variable that is both 1) not in the consumer's choice set and 2) is homogenous of degree zero in money denominated variables. This implies that we could potentially achieve a good fit of the money demand function using additional parameters, without damaging the micro foundations of the model.

Theorem 2 states that the parameter ϕ, representing the frequency of observation considered in the analysis must be included for the model to be admissible, the only exception arises if $\gamma = 1$ which, in the limit, corresponds to the Cobb-Douglas MIUF functional form. For the Cobb-Douglas utility function, money demand depends upon the ratio of y to i. Since income and interest return are both flow variables the frequency terms 'cancel out'.

<Theorem 2 Here>

Finally, the Cobb-Douglas MIUF functional form, and the commonly used form monotonic transformation of the Cobb-Douglas, $U\left(c, \frac{m}{p}\right) = \ln c + \theta \ln \frac{m}{p}$, is inadmissible due to the third criterion of admissibility because it implies that the interest elasticity of money demand is -1, which is outside of the range of values that is empirically observed. Mis-specifying the interest elasticity of money demand, $-\gamma$, causes the income velocity of money, the demand for real money balances, and the welfare cost of inflation to also be mis-specified. The attribute that keeps the Cobb-Douglas utility function from violating the second criterion of admissibility, its prediction that the money demand depends upon the ratio of y to i, is precisely the attribute that renders the functional form unattractive. Whereas an

[6] The linear utility function, which is a member of the CES class, could provide non-monetary equilibria, however, its prediction of a (negative) infinite interest elasticity of money demand renders this functional form inadmissible.

analysis of the Exchange Equation implies that the *income* elasticity of money demand is exactly 1 for this model, there is no corresponding rule governing the interest elasticity of money demand; an *a priori* imposition of -1 for the interest elasticity of money demand is neither supported by theory or data.

2. 2. A Dynamic Baumol CIA Model (Model b)

For a CIA model, and accompanying admissible functional form, I propose the following time-dynamic and general equilibrium extension of Baumol's (1952) shoe-leather model of money velocity and demand. There is a real transactions cost, τ, that reduces the goods available for consumption. Consider this as a shoe-leather expenditure of transacting as in Baumol (1952) and Tobin (1956). The frequency of transactions in which money obtained is viewed as endogenous and the length of the transactions period is unrelated to the length of the period considered by the analyst. Since transactions are costly, the consumer must balance the cost of transacting against the opportunity cost of holding money. The frequency of the consumer's transactions may be greater than or less than the frequency of data utilized by the analyst.

The introduction of a 'transactions production function' is a contribution of this paper. Theorem 3 proves that such a function must be homogenous of degree zero to be admissible. Theorem 4 shows that there is an exact equivalence between the MIUF and CIA frameworks; specifically, for any admissible functional form for utility in model *a* there is a corresponding admissible 'transactions production function' in model *b* and vice versa.

The consumer's budget constraint, expressed in real terms, is:

$$p(c+\tau)+\omega m' = py + m + \omega - 1. \tag{1b}$$

where τ represents the resources used up in transacting and CIA constraint is:

$$py = m \cdot n(\tau) \tag{2b}$$

where $n(\cdot)$ is the function representing the number of transactions in the period of analysis. Since consumption is always valued, equations 1b and 2b will hold as an equality in equilibrium. If we compare the CIA constraint, equation 2b, to the Exchange Equation we conclude that n represents the velocity of money. The most basic CIA model fixes n at one, no matter the frequency of data considered in the analysis or definition of money utilized. This section treats n analogously to Baumol's (1952) paper; namely more transactions may be obtained, real money balances thus conserved, by expending resources.

The number of transactions per period, n,[7] is not constrained to be an integer because the consumer's transaction period is not required to line up evenly with the frequency with which the data is reported or the model analyzed. For instance, if the data is reported monthly and the consumer obtains cash every three weeks, then the monthly velocity will be *about*[8] 0.75. The prevalent idea that the number of transactions is constrained to be an integer, Corbae (1993) and Rodríquez Mendizábal (2006), seems to arise from the implicit assumption that there is always a transaction occurring at the beginning of the researcher's period of analysis.

Neither is the number of transactions constrained to be at least one in this study, as in Lucas and Stokey (1987) and studies utilizing their structure. The consumer is viewed to be unaware of, and uninterested in, the frequency with which the researcher observes him, and he does not receive a 'free' transaction at the beginning of the researcher's observation period, as in all of the previous CIA models. The notions of 'transactions period' and 'period of analysis' should be analytically separate.

Finally, this model differs from standard CIA models because the CIA constraint is placed upon *equilibrium* spending, y, rather than the *consumer's choice* level of spending, c. Placing c in the CIA constraint makes it difficult to obtain a closed-form solution for this problem.

The market-clearing conditions are:

$$\begin{aligned} m' &= m = 1 \\ c + \tau &= y \end{aligned} \tag{3b}$$

and the representative consumer's value function is:

$$v(S) = \max_{c,\tau,m'} \{U(c) + \beta \cdot Ev(S')\} \tag{4b}$$

where the utility function is a CRRA representation:[9]

$$U(c) = \frac{1}{\phi(1-\alpha)} \cdot (\phi c)^{1-\alpha}, \quad \alpha \neq 1 \ . \tag{5b}$$

Note again that the ϕ term annualizes the flow of consumption within the observation period in equation 5b and then reduces the weight of the 'observation period' utility according to the

[7] The function n is the frequency with which the consumer obtains the initial level of cash balances. If we think of the household as a worker-shopper pair, as in Lucas (1980), then a 'transaction' is the occasion in which the shopper replenishes his or her cash balances by visiting the family shop, settling accounts, and collecting income payments in the form of cash to use in future purchasing in other shops.

[8] 'About' is in italics because there are not exactly four weeks per month.

[9] We will see that the degree of relative risk aversion has no effect on the derived unit-free and observable measures. In future studies it may be possible to use this observation to separate the monetary aspects of a model from other implications to simplify the relevant analysis.

frequency of observation considered in analysis. This is required to receive consistent answers from studies using different frequency of analysis.

The consumer seeks to maximize the value function subject to the budget and CIA constraints. The first-order conditions for the consumer's problem are given below, where λ and μ are the Lagrange multipliers for the budget and CIA constraints, respectively:

$$v_c(S) = U_c(c) - p\lambda = 0 \tag{6b}$$

$$v_\tau(S) = m\,n_\tau\,\mu - p\lambda = 0 \tag{7b}$$

$$v_{m'}(S) = -\omega\lambda + \beta E v_{m'}(S') = 0 \tag{8b}$$

and the term $v_{m'}(S')$ is found using the envelope condition:

$$E v_{m'}(S') = E[\lambda' + n'\mu'] \ . \tag{9b}$$

Again, analysis is restricted to the risk-free steady-state equilibrium with constant output and a constant money growth rate, $y' = y$ and $\omega' = \omega$.

Combining the first order conditions, equations 6b-9b, with the market clearing conditions, equation 3b, the CIA constraint, equation 2b, the steady-state conditions, and interest rate definition, equation 10a, yields the following Euler equations:

$$\lambda = \frac{y}{n} \cdot U_c(y) \tag{10b}$$

$$\mu = \frac{U_c(y)}{n_\tau} \tag{11b}$$

$$\mu = \frac{i}{n} \cdot \lambda \ . \tag{12b}$$

The first equation above combines equations 2b, 3b and 6b, the second 3b, 6b and 7b, and the third combines 8b and 9b. Finally, these three equations are combined to yield an expression governing the equilibrium shoe-leather expenditure:

$$n^2 = n_\tau\, i\, y \tag{13b}$$

This is a generalization of Baumol's famous square root formula for the derived velocity of money.

With the solution of $n(\tau)$ in hand we could solve for the equilibrium price level, *p*, using equation 2b. As with the MIUF model, the equilibrium *p* allows us to define the consumer's income velocity of money and demand for real money balances. Theorem 3, introduced below, shows that the function *n* must be homogenous of degree zero in the money denominated variables, τ and *y*, for the income velocity of money and money demand function to be homogenous of degree zero and one, respectively. I therefore propose the following functional form for the transactions production function:

$$n(\tau) = \frac{\theta}{\phi} \cdot \left(\frac{\tau}{y}\right)^{\psi} \quad . \tag{14b}$$

Theorem 4 implies that this form is, in fact, a very general representation of the admissible forms for this problem. The number of transactions depends positively upon the ratio of shoe-leather expenditure, τ, to the volume of spending, y. The parameters ψ and θ represent the marginal and total factor productivity of shoe-leather in producing transactions, respectively. Inclusion of the fixed parameter ϕ, representing the frequency of data observations per year, is needed to maintain admissibility of the functional form across studies of differing frequencies; as stated before, we expect the number of transactions in a quarterly analysis to be ¼ the number in an annual analysis, so $\phi = 1$ in an annual study and $\phi = 4$ in a quarterly one.

From equation 13b and 14b we obtain a closed-form solution for τ, this is the indirect transactions production function under optimal behavior:

$$\tau = \left(\frac{\psi}{\theta} \cdot \phi i\right)^{\frac{1}{1+\psi}} \cdot y \quad . \tag{15b}$$

As in Baumol (1952), shoe-leather expenditure depends upon positively upon both y and i.

Combining equations 2b, 14b, and 15b yields the equilibrium price level:

$$p = \frac{1}{\phi y} \cdot \theta^{\frac{1}{1+\psi}} \cdot (\psi \phi i)^{\frac{\psi}{1+\psi}} \quad . \tag{16b}$$

To demonstrate that there is a strict degree of equivalence between this CIA function and the admissible MIUF model, I rewrite the equilibrium pricing function as:

$$p = \frac{\tilde{\theta}}{\phi y} \cdot (\phi i)^{\gamma} \quad . \tag{17b}$$

where $\gamma \equiv \dfrac{\psi}{1-\psi}$ and $\tilde{\theta} \equiv \theta^{\frac{1}{1+\psi}} \cdot \psi^{\frac{\psi}{1+\psi}}$ are exogenous parameters. Then the annualized velocity of money is:

$$V \equiv \frac{P \phi y}{\overline{M}} = p \phi y = \tilde{\theta} \cdot (\phi i)^{\gamma} \tag{18b}$$

and the level of real balances is:

$$L \equiv \frac{M}{P} = \frac{1}{p} = \tilde{\theta}^{-1} \cdot (\phi i)^{-\gamma} \cdot \phi y \quad . \tag{19b}$$

The income elasticity of money demand is again one and the interest elasticity is:

$$\varepsilon_{Li} = \frac{\partial L}{\partial i} \cdot \frac{i}{L} = -\gamma \ . \tag{20b}$$

The interest elasticity of money demand, $-\gamma$, is related to the consumer's elasticity of substitution between *c* and *L*; in model *a* this is a direct result of the utility function, in model *b* this arises indirectly from the transactions technology.

Equations 17b through 20b are identical to their MIUF model counterparts. While a degree of equivalence between MIUF and CIA models has been noted before by Feenstra (1986) and Wang and Yip (1992), imposing the concept of admissibility upon functional forms makes the equivalence exact. Theorem 4 demonstrates that for any admissible utility function in model *a* there is a corresponding transactions function in model *b* that yields identical solutions for the observable variables and for Bailey's measure of the welfare cost of inflation; the reverse is also true.

Finally, the indirect utility function is derived by substituting the optimal shoe leather expenditure, equation 15b, into the utility function, equation 5b:

$$U^*(i, y) = \frac{1}{1-\alpha}\left[\phi y - \phi\tau(i, y)\right]^{1-\alpha} \ . \tag{21b}$$

The first unit-free measure of the welfare cost of inflation is the proportional increase in output required to compensate the consumer for living with inflation:

$$\Pi(i) \equiv \frac{\hat{y}}{y} - 1 = \frac{\tau(i, y)}{y} \ . \tag{22b}$$

The amount of income required to compensate the consumer for living with inflation is precisely the amount of she-leather expended, τ.

Here, again, the optimal inflation rate is where i = 0, and the welfare cost of inflation grows with the inflation and, thus, nominal interest rates. Equation 22b is similar in form to the welfare cost of inflation function for the MIUF model, equation 18a, but is not identical. In model *a* inflation reduces utility by reducing the level of real money balances, in model *b* consumption is reduced by the shoe-leather cost, reducing utility. Since the observable implications for both models are the same there is no way to decide between the models on an empirical basis. For this model the second measure of the welfare cost of inflation, based upon Bailey (1956), is identical to equation 19a from the MIUF model, and these are both a scalar multiple of equation 22b. Bailey's measure of the welfare cost of inflation is conceptually preferable to the measure based upon inflation compensating level of income because it yields identical results for models with identical implications for the observable variables; the cost measure does not depend arbitrarily on the modeling decision.

Theorem 3 proves that the model's predictions for the unit-free measures do not depend upon the unit of account if and only if the function n is homogenous of degree zero in money denominated variables. The existence of the transactions production function, and the derivation of its properties is a contribution to the CIA literature.

<Theorem 3 Here>

As with the MIUF model, the CIA model satisfies the second condition of admissibility if and only if the parameter ϕ represents the researcher's frequency of observation; the sole exception being when $\gamma = 1$. If $\gamma = 1$ the demand for real money balance is always proportional to the ratio of y to i, so the frequency of analysis terms 'cancel out'. Since this case implies an interest elasticity of -1 it does not satisfy the third requirement of admissibility, that the model's parameters be consistent with observation.

The most commonly used CIA functional form in use today is the 'cash-credit goods' model introduced by Lucas and Stokey (1987). In this model the CIA constraint applies to a subset of total purchases. This allows the velocity of money to vary in value, taking on values greater than or equal to one. Inflation is important in this model because it drives a tax wedge between the marginal utilities of consumption for the cash and credit goods. The cash-credit model is not admissible because the unit-free measures vary with the frequency of data that the researcher selects for the analysis. For example in their study of the welfare cost of inflation Cooley and Hansen (1989, p. 743) report that their unit-free measure of the welfare cost of inflation is more than three times larger for a monthly model than a quarterly model at all interest rates. Additionally, the requirement that the income velocity of money be at least one, no matter the frequency of data or measure of money constitutes a violation of the second condition of admissibility. Cooley and Hansen (1991, p. 492) cite this limitation as the reason for constructing a new monetary data series, 'M1 held by households' for their study of the welfare effect of inflation. Overall, there is no way to map the predictions for the unit-free measures of the cash-credit of one frequency into a study of another frequency, as a result the model's predictions depend arbitrarily on this choice and the model is not internally consistent.

Theorem 4 proves that for any admissible utility function in model a there is a corresponding transactions production function in model b that yields identical predictions for the observable unit-free measures and for Bailey's measure of the welfare cost of inflation, and vice versa. This has a number of implications: first, it does not matter which framework, MIUF or CIA, the researcher utilizes. Second, we cannot ascribe other motives, besides the transactions motive, for money's inclusion in the utility function of a MIUF model, unless we can also find such a motive within the CIA framework. If we wish to study the effect of other motives for holding money then we must incorporate such a motive into the model, for instance see Imrohoroglu's (1992) study of the role of real balances as insurance.

Another byproduct of Theorem 4 is that the derived money demand function for these simple MIUF and CIA models must be of the log-log form, $L = A(\phi i)^{-\sigma} \cdot y$; see equation 6 in Theorem 4. The money demand function may not, for instance, be of the semi-log form. The semi-log money demand function is not consistent with the utility maximization problems inherent in models *a* and *b*. Lucas (2000, p. 250) points out that this implies that seigniorage revenue is always increasing in the money growth rate. Lucas (2000) also demonstrates that this choice of form has strong implications for the welfare cost of inflation. This is another example of the concept of admissibility placing strong restrictions upon the theoretical content of a model.

3. Conclusion

The concept of admissibility requires the structure of a monetary model to conform to some basic microeconomic principles, for instance the model's results should be invariant to the unit of account and frequency of measurement. A number of modeling decisions in monetary macroeconomics, for instance 1) the measure of money to use in the analysis, 2) the frequency of data to utilize in an analysis,[10] 3) whether to utilize the MIUF or CIA model in the analysis, have long appeared to be arbitrary on one hand, but central to the models' predictions on the other. Requiring a model to conform to the rules of admissibility appears to render each of these choices as moot. Similarly, the selection of the functional form for money demand for use in theoretical and empirical studies has been at once important yet seemingly arbitrary; admissibility implies that only one form is consistent with utility maximization in the simple general equilibrium models considered.

The finding of a one-to-one equivalence between admissible MIUF and CIA models is an important finding. It is comforting that important macroeconomic predictions do not depend upon the modeling structure used to generate the prediction.

[10] The measure of money utilized and the frequency of data used *may* be important, but they are not important because of the relative size of the stock of money to the flow of purchases in the period of analysis.

Literature:

Arnwine, Neil. 2007. "'Shoe-Leather' and 'Bricks-and-Mortar' as Inputs into Transacting." Manuscript.

Arnwine, Neil and Yigit, Taner M. 2007. "What Fisher Knew About His Relation, We Sometimes Forget." Bilkent University Economics Department Working Paper.

Bailey, Martin J. 1956. "The Welfare Cost of Inflationary Finance." *The Journal of Political Economy* 64(2):93-110.

Bewley, Truman. 1983. "A Difficulty With the Optimum Quantity of Money." *Econometrica* 51(5):1485-1504.

Baumol, William. 1952. "The Transactions Demand for Cash: An Inventory-theoretic Approach." *Quarterly Journal of Economics*, 66:545-556.

Brock, William A. "Money and Growth: The Case of Long Run Perfect Foresight." *International Economic Review* 15(3):750-777.

Clower, Robert W. 1967. "A Reconsideration of the Micro foundations of Money." *Western Economic Journal* 6(1):1-6.

Cooley, Thomas F. and Hansen, Gary D. 1989. "The Inflation Tax in a Real Business Cycle Model." *American Economic Review* 79(4):733-748.

Cooley, Thomas F. and Hansen, Gary D. 1991. "The Welfare Costs of Moderate Inflations." *Journal of Money, Credit and Banking* 23(3):483-503.

Corbae, Dean. 1993. "Relaxing the Cash-in-Advance Constraint at a Fixed Cost." *Journal of Economic Dynamics and Control* 17:51-64.

Feenstra, Robert C. 1986. "Functional Equivalence Between Liquidity Costs and the Utility of Money." *Journal of Monetary Economics*, 17(2):271-291.

Fischer, Stanley. 1979. "Capital Accumulation on the Transition Path in a Monetary Optimizing Model." *Econometrica* 47(6):1433-1439.

Fisher, Irving. 1896. "Appreciations and Interest." *Publications of the American Economic Association* 11(4):1-98.

Freeman, Scott and Kydland, Finn E. 2000. "Monetary Aggregates and Income." *American Economic Review* 90(5):1125-1135.

Friedman, Milton. 1969. "The Optimum Quantity of Money." *In The Optimal Quantity of Money and Other Essays*. 1-50, Chicago: Aldine.

Imrohoroglu, A. 1992. "The Welfare Costs of Inflation under Imperfect Insurance." *Journal of Economic Dynamics and Control* 16(1):79-91.

Lucas, Robert E., Jr. 1980. "Equilibrium in a Pure Currency Economy." *Economic Inquiry* 18:203-220.

Lucas, Robert E., Jr. and Stokey, Nancy L. 1987. "Money and Interest in a Cash-in-Advance Economy." *Econometrica* 55(3):491-513.

Marshall, Alfred. 1885. "The Graphic Method of Statistics." *Journal of the Royal Statistical Society*, 50:251-60.

Obstfeld, Maurice and Rogoff, Kenneth. 1986. "Ruling Out Divergent Speculative Bubbles." *Journal of Monetary Economics* 17(3):340-362.

Phelps, Edmund S. 1973. "Inflation in the Theory of Finance." *Swedish Journal of Economics* 75(1):67-82.

Rodríguez Mendizábal, Hugo. 2006. "The Behavior of Money Velocity in High and Low Inflation Countries." *Journal of Money, Credit and Banking* 38:209-228.

Sidrauski, Miguel. 1967. "Rational Choice and Patterns of Growth in a Monetary Economy." *American Economic Review* 57:534-544.

Svensson, Lars E.O. 1985. "Money and Asset Prices in a Cash-in-Advance Economy." *Journal of Political Economy* 93:919-944.

Tobin, James. 1956. “The Interest Elasticity of the Transactions Demand for Cash.” *Review of Economics and Statistics* 38:2412-247.
Wang, Ping and Yip, Chong K. 1992. “Alternative Approaches to Money and Growth.” *Journal of Money, Credit, and Banking* 24:553-562.

Theorem 1: The MIUF model described by equations 2a-4a has unit-free measures that are homogenous of degree zero in money denominated variables if and only if the utility function is an affine transformation of a CES utility function.

Proof:

i) "A CES utility function, and affine transformations thereof, yield unit-free measures that are homogenous of degree zero in money denominated variables."

The analysis of equations 15a, 19a, 20a, and 21a demonstrates that one particular affine transformation of the CES utility function, the CRRA specification, is admissible. This is clearly also true for the CES utility function obtained by setting $\alpha = 0$.[11] Any affine transformation of this CES utility function will obtain the same pricing function in equation 12a because it will yield the same ratio of marginal utilities, therefore we obtain the same solutions for money velocity, money demand, and money demand elasticities.

Since the CES utility indirect utility function is linearly homogenous, any affine transformation of this will also be a linearly homogenous function. Therefore the derived unit-free measure of the welfare cost of inflation will be homogenous of degree zero.

ii) "An admissible utility function must be of the CES class."

a) An admissible utility function must yield a money demand function of the form: $L = f(i) \cdot y$, since it must be homogenous of degree one in the only money-denominated variable. This is necessary and sufficient for the measures of the income velocity of money and the income and interest elasticities of money demand to be homogenous of degree zero. To show that the measure of the welfare cost of inflation is homogenous of degree zero in *y*, I calculate the indirect utility function:

$$U^*(i, y) = \max_{c,L} U(c, L) = U(y, f(i) \cdot y) \tag{1}$$

where U is the utility function in consumption and real balances and U^* is the indirect utility function. We can not guarantee *a priori* that $U^*(0, y)$ is finite, so the unit-free measure of the welfare cost of inflation used here is based upon the interest elasticity of inflation compensating income. This is found by setting the total derivative of the indirect utility function equal to zero:

$$d\,U^*(i, y) = \left[U_1(y, f(i) \cdot y) + f(i) \cdot U_2(y, f(i) \cdot y)\right] dy + y f_i(i) \cdot U_2(y, f(i) \cdot y)\, di = 0 \tag{2}$$

The welfare cost of inflation is expressed as the interest elasticity of compensating income calculated from equation 2:

[11] Since output is finite we are not concerned about the problem of existence of a solution due to the fact that U is linearly homogenous when $\alpha = 0$.

$$\varepsilon_{yi} \equiv \frac{dy}{di} \cdot \frac{i}{y}\bigg|_{dU=0} = -\frac{i \cdot f_i(i) \cdot U_2(y, f(i) \cdot y)}{U_1(y, f(i) \cdot y) + f(i) \cdot U_2(y, f(i) \cdot y)}$$
$$= -\frac{i \cdot f_i(i)}{f(i) + \frac{U_1(y, f(i) \cdot y)}{U_2(y, f(i) \cdot y)}} \tag{3}$$

This welfare cost is a unit-free measure if and only if the ratio of the marginal utilities under the optimal behavior does not vary with income. This is true if and only if the utility function is in the CES family, or if it is an affine transformation of a CES function.

b) Now I will show that if the derived demand function is of the form: $L = f(i) \cdot y$, then the utility function must be an affine transformation of a CES utility function. The degree of elasticity of substitution is calculated as follows:

$$\sigma \equiv \frac{d\,\frac{L}{c}}{d\,\frac{P_L}{P_c}} \cdot \frac{\frac{P_L}{P_c}}{\frac{L}{c}}$$
$$= \frac{d\ f(i)}{d\,i^{-1}} \cdot \frac{i^{-1}}{f(i)} \tag{4}$$
$$= -i \cdot \frac{f'(i)}{f(i)}$$

where the relative price of L to c, $\frac{P_L}{P_c}$, is given by i^{-1}. The 'price' of the money stock relative to the price of consumption, a flow, is given by the discounted present value of the utility services of money into the infinite future, which is the inverse of the nominal interest rate. The elasticity of substitution between real balances and consumption is constant with respect to money denominated variables.▪

Theorem 2: The MIUF model described by equations 2a - 4a with $\gamma \neq 1$ has derived unit-free measures V, ε_{Ly}, ε_{Li}, and $\varepsilon_{\hat{y}i}$ that are invariant with respect to the frequency of data used in analysis if, and only if, the fixed parameter ϕ equals the frequency of data.

Proof:

i) "If ϕ represents the frequency of data considered in the analysis, then we observe that each of the derived unit-free measures is homogenous of degree zero in $\{\phi, i, y\}$."

This is obvious from equations 15a, 17a, 19a, and 20a.

ii) "If ϕ does not represent the frequency of data considered in the analysis then at least one of the unit-free measures varies with the frequency of the data, unless $\gamma = 1$."

If $\gamma \neq 0$ and $\gamma \neq 1$ it is apparent that the term $(\phi \cdot i)$ in equation 20a varies with the frequency of analysis unless ϕ is equal to the number of observations per year. The utility function is not defined for $\gamma = 0$, however in the limit as γ approaches zero, the utility function converges to the Leontief form. In this case, the level of real money balances is in a fixed proportion to income, no matter the frequency of analysis so, therefore, the predicted annual velocity of money varies with the frequency of analysis. For all values of γ, excepting $\gamma = 1$, failure to have ϕ represent the frequency of data used in the analysis causes at least one unit-free measure to vary with the frequency of analysis.

Theorem 3: The CIA model described by equations 1b-5b has unit-free measures that are homogenous of degree zero in money denominated variables if and only if the 'transactions production function', $n(\tau)$, is homogenous of degree zero in $\{\tau, y\}$.

Proof:

i) "$n(\tau)$ is homogenous of degree zero in $\{\tau, y\}$ iff the unit-free measures, except for the welfare cost of inflation, are homogenous of degree zero in y."

By equations 2b and 3b $n(\tau)$ is homogenous of degree zero in $\{\tau, y\}$ iff p^{-1} is homogenous of degree one in $\{\tau, y\}$. Therefore $L \equiv \frac{M}{P} = \frac{1}{p}$ is homogenous of degree one in y iff $n(\tau)$ is homogenous of degree zero in $\{\tau, y\}$. This implies that the interest and income elasticities of money demand are homogenous of degree zero iff $n(\tau)$ is homogenous of degree zero in $\{\tau, y\}$. Since $n(\tau)$ is the velocity of money, velocity is homogenous of degree zero iff $n(\tau)$ is homogenous of degree zero.

ii) "There is a unit-free measure of the welfare cost of inflation that is homogenous of degree zero in y iff $n(\tau)$ is homogenous of degree zero in $\{\tau, y\}$."

a) That $\frac{\tau(i, a \cdot y_1)}{a \cdot y_1} = \frac{\tau(i, y_1)}{y_1}$ implies $n\left(\frac{\tau(i, a \cdot y_1)}{a \cdot y_1}\right) = n\left(\frac{\tau(i, y_1)}{y_1}\right)$ for $a > 0$ is obvious, the first term is a unit-free of the welfare cost of inflation, the second is the transactions production function.

b) If $n(\tau)$ is homogenous of degree zero in $\{\tau, y\}$ then by part (i) the derived money demand function is homogenous of degree one and can be written as $L(y, i) = f(i) \cdot y$. Then clearly Bailey's (1956) measure of the welfare cost of inflation, introduced in equation 20a, can be written as $\Omega(i) = \frac{-1}{y} \cdot \int_0^{\phi i} L(y, i)\, di = \int_0^{\phi i} f(i)\, di$. This is clearly homogenous of degree zero in money denominated variables.

Theorem 4: For any admissible utility function in model *a* there is a corresponding admissible transactions production function, $n(\tau)$, in model *b* that yields identical solutions for the observable unit-free variables, *V*, ε_{Ly}, ε_{Li}, and for Bailey's measure of the welfare cost of inflation. For each admissible functional form for $n(\tau)$ in model *b* there is a corresponding admissible utility function in model *a*.

<u>Proof</u>:

Any admissible function in models *a* or *b* will have a derived money demand of the form: $L = f(i) \cdot y$.

i) For any admissible utility function in model *a* there is a corresponding admissible transactions production function, $n(\tau)$, in model *b*.

a) Let $U\left(c, \frac{m}{p}\right)$ be an admissible MIUF model utility function with a CES parameter of σ. We can derive the corresponding $f(i)$ for this model by solving the first order homogeneous differential equation with a variable term implicit in equation 4 of Theorem 1. Rewriting equation 4 we obtain:

$$f'(i) + \frac{\sigma}{i} \cdot f(i) = 0 \tag{5}$$

Solving for *f* we obtain:

$$f(i) = Ai^{-\sigma} \tag{6}$$

where *A* is a constant of integration. *A* corresponds to the parameter $\tilde{\theta}^{-1}$ in the derived money demand functions, equations 16a and 19b. Recall that σ is the negative of the interest elasticity of money demand, so $\sigma \equiv \gamma$. Given σ and *A* we can find the model's $f(i)$ and then each of the observable variables may be derived and Bailey's measure of the welfare cost of inflation is found by integrating over $f(i)$.

b) For any *A* and σ corresponding to the admissible MIUF utility function *U* we can obtain a corresponding admissible functional form for *n* that yields identical predictions for the derived observable variables of real money balances, income velocity of money and money demand elasticities and for Bailey's measure of the welfare cost of inflation.

Begin with the transactions production function defined in equation 14b. For simplicity let's re-label this as equation 7:

$$n(\tau) = \frac{\theta}{\phi} \cdot \left(\frac{\tau}{y}\right)^{\psi} . \tag{7}$$

For this model we obtained $f(i) = \tilde{\theta}^{-1}(\phi i)^{-\gamma}$ where, for simplicity, I defined $\gamma \equiv \frac{\psi}{1-\psi}$ and $\tilde{\theta} \equiv \theta^{\frac{1}{1+\psi}} \cdot \psi^{\frac{\psi}{1+\psi}}$. Recall that the interest elasticity of money demand is determined by the parameter γ, i.e. $\gamma = \sigma$, and that $A = \tilde{\theta}^{-1}$ from part i(a) of this proof. Eliminating the parameters γ, θ and ψ from equation 7 by substitution, we obtain:

$$n(\tau) = \frac{\tilde{\theta}}{\phi} \cdot \left(\tilde{\theta} \cdot \frac{1+\gamma}{\gamma} \cdot \frac{\tau}{y} \right)^{\frac{\gamma}{1+\gamma}} . \tag{8}$$

The CIA model defined by equations 1b-5b, together with equation 8, is identical in its predictions to MIUF model.

ii) Let n be an admissible transactions production in model b, there is an admissible utility function in model a that yields identical predictions for the observable variables and for Bailey's measure of the welfare cost of inflation.

Since n is admissible then there is a function f such that $L = f(i) \cdot y$. Furthermore, from part i(a) of this proof, f must be of the form $f(i) = Ai^{-\sigma}$ where σ is the elasticity of substitution between consumption and the level of real money balances, since f is derived from model b.

If a corresponding functional form in model a exists, Theorem 1 states that the utility function must be CES and Theorem 2 states that the utility function must correct for the frequency of analysis. Therefore, if a corresponding utility function in model a exists, it must be of the form of equation 13a, or a monotonic transformation of this equation.

Equations 2a-4a, together with equation 13a, defines model a so we obtain all of its results. Specifically, equation 16a implies that $f(i) = \tilde{\theta}^{-1}(\phi i)^{-\gamma}$. Given an arbitrary admissible transactions production function in model b with derived parameters A and σ we obtain identical predictions from model a with the utility function defined in equation 13a with parameters $\gamma = \sigma$ and $\theta = \tilde{\theta}^{-\frac{1}{\gamma}} = A^{-\frac{1}{\gamma}}$.

Appendix 1: List of parameters and variables

- α – Degree of relative risk aversion.
- β – Discount rate.
- ϕ – Number of observations per period – the data frequency.
- γ – Elasticity of substitution in model *a*.
- λ – Lagrange multiplier for the consumer’s budget constraint.
- μ – Lagrange multiplier for the consumer’s cash-in-advance constraint in model *b*.
- θ – Average velocity of money; in utility function in *a* and production function in *b*.
- σ – General measure of elasticity of substitution between consumption and money.
- τ – Shoe leather expenditure in model *b*.
- ω – Money growth rate plus one.
- ψ – Productivity of shoe-leather in creating transactions in model *b*.
- c – Consumption.
- i – Nominal interest rate.
- n – Transactions production function in model *b*.
- p – Deflated price level.
- v – Value function.
- y – Income.
- A – Constant of integration.
- E – Expectations operator.
- L – Derived money demand function.
- M – Nominal money stock.
- P – Nominal goods price level.
- S – Set of state variables.
- U – Utility function.
- V – Income velocity of money.