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Monotonic Extension

By Farhad Husseinov

Abstract

The main result of this paper gives a necessary and sufficient condition for a continuous, strictly monotone function defined on a closed set of a Euclidean space to be extendible to the whole space.

Key words: Monotonic function, extension, normally ordered topological spaces, selection.

Math Subject Classification: 26A48, 54C30.

1. Introduction

It is somewhat surprising that the problem of extending a continuous monotonic function defined on a subset of a Euclidean space into the entire space with preservation of its properties has received scant attention. In this paper we tackle this problem for the cases of both monotonic and strictly monotonic functions. For each case we find a property that is necessary and sufficient for the given function to be extendible by preserving the continuity and strict monotonicity properties. However, the central result of this paper (Theorem 3) deals with the case of strictly monotone functions.

Nachbin [1] studied the problem of extending a continuous, (weakly) monotone (*isotone* in his terms) and bounded functions defined on closed subsets of an arbitrary normally ordered topological space. These spaces, introduced in [1], generalize normal topological spaces to the spaces equipped with an order relation. He discovered a property (called further the *Nachbin property*) that is necessary and sufficient for the existence of an extension that satisfies the said properties [1, Theorem 2]. Nachbin's extension theorem has found applications in diverse fields.

In this paper, we first introduce a property that proves to be equivalent to the Nachbin property.and hence we get a modification of Nachbin's extension theorem. However, we give an independent short proof of this version based on Michael's selection theorem [2] for the case of Euclidean spaces ordered with the standart componentwise order. We assume that this proof will provide intuitions about difficulties in treating the case of strictly increasing functions.

It seems impossible to formulate a version of the Nachbin property that will be suitable for treating strictly increasing functions. However, the property introduced here readily strengthens for this case.

2. Notation and Preliminaries

A *preorder* $\succcurlyeq$ on a set X is a reflexive ans transitive binary relation on this set. If in addition, $\succcurlyeq$ is antisymmetric (that is $x \succcurlyeq y$ and $y \succcurlyeq x$ imply $x = y$) then it is called an *order.* A real function f defined on a subset D of the preordered set X is said to be *increasing* if for any two points $x, y \in D$ such that $x \succcurlyeq y$, we have $f(x) \geqslant f(y)$. In turn f is said to be *strictly increasing* if for any two points $x, y \in D$ such that $x \succ y$, we have $f(x) > f(y)$. Decreasing and strictly decreasing functions are defined in a similar way. A function $f' : D' \to R$ is an *extension* of a function f if $D \subset D'$ and $f'(x) = f(x)$ for all $x \in D$.

A set A in X is *decreasing* if $x \in A$ and $x \succcurlyeq y$ imply $y \in A$. An *increasing* set is defined dually. A set X equipped with the both topology τ and preorder $\succcurlyeq$ is said to be *normally preordered* if, for every closed disjoint subsets F_0 and F_1 of X, such that F_0 is decreasing and F_1 is increasing, there exist two disjoint open sets U_0 and U_1 of X, such that U_0 is decreasing and contains F_0 and U_1 is increasing and contains F_1.

Let $(X, \tau, \succcurlyeq)$ be an arbitrary preordered topological space. The *decreasing closure*, denoted as $\mathcal{D}(A)$, of a set A in R^n is the smallest decreasing and closed set containing A. *The increasing closure* of A is defined dually and denoted as $\mathcal{I}(A)$. For a function $f : D \to R$ and a real α, set

$$L_f(\alpha) = \{x \in D : f(x) \leqslant \alpha\} \text{ and } U_f(\alpha) = \{x \in D : f(x) \geqslant \alpha\}.$$

The Nachbin property reads as follows: for each $\alpha, \alpha' \in R$ such that $\alpha < \alpha'$

$$\mathcal{D}(L_f(\alpha)) \cap \mathcal{I}(U_f(\alpha')) = \emptyset.$$

We now introduce a condition that is equivalent to the Nachbin condition. Denote by $\mathcal{V}_d^x$ and $\mathcal{V}_i^x$ the collections of open decreasing and open increasing sets containing x. For a given function $f : D \to R$, where D is an arbitrary set in X, we set

$$m_f(x) = \inf_{V_1^x \in \mathcal{V}_d^x} \sup\{f(z) : z \in D \cap V_1^x\} \text{ and } M_f(x) = \sup_{V_2^x \in \mathcal{V}_i^x} \inf\{f(z) : z \in D \cap V_2^x\},$$

with the agreement that $m_f(x) = \inf\{f(z) : z \in D\}$ and $M_f(x) = \sup\{f(z) : z \in D\}$, if $D \cap V_1^x = \emptyset$ for some $V_1^x \in \mathcal{V}_d^x$ and $D \cap V_2^x = \emptyset$ for some $V_2^x \in \mathcal{V}_i^x$, respectively. Further we will omit subindex f in the notations m_f and M_f in cases where it is clear which function is refered to.

We will see shortly that the Nachbin property is equivalent to the following property:

$$m_f(x) \leqslant M_f(x) \text{ for all } x \in X. \tag{1}$$

Before we give some examples illustrating property (1) and its strengthening (see property (13) below).

Examples: Set $D_+ = \{(x,y) \in R^2 | xy = -1, y > 0\}$, $D_- = -D_+$, and $D = D_+ \cup D_-$. Define functions $f_i : D \to R$ as $\frac{y}{1+y}$ on D_+ and $i - 2 + \frac{y}{1-y}$ on D_- for $i = 1, 2, 3$.

It is easy to see that function f_1 has no increasing extension, f_2 has an increasing extension but not a strictly increasing extension, and f_3 has a strictly increasing extension into R^2. We have $m_{f_1}(0) = m_{f_2}(0) = m_{f_3}(0) = 0$ and $M_{f_i}(0) = i - 2$ for $i = 1, 2, 3$. Hence $m_{f_1}(0) > M_{f_1}(0)$, $m_{f_2}(0) = m_{f_2}(0)$, and $m_{f_3}(0) < M_{f_3}(0)$. Thus property (1) is violated for function f_1 and satisfied for functions f_2 and f_2 at point $x = 0$.

Claim: *The Nachbin property and property (1) are equivalent.*

Proof: Indeed, let $m(x) > M(x)$ for some $x \in X$. It follows from the definitions of functions m and M that for each $V_1^x \in \mathcal{V}_d^x$ and $V_2^x \in \mathcal{V}_i^x$

$$\sup\{f(z) : z \in D \cap V_1^x\} \geqslant m(x) > M(x) \geqslant \inf\{f(z) : z \in D \cap V_2^x\}.$$

This implies $x \in \mathcal{D}(L_f(M(x))) \cap \mathcal{I}(U_f(m(x)))$, which contradicts the Nachbin property. Assume the Nachbin property is violated, that is, there exist reals α, α' with $\alpha < \alpha'$ such that

$$\mathcal{D}(L_f(\alpha)) \cap \mathcal{I}(U_f(\alpha')) \neq \emptyset.$$

Let x belong to this intersection. Then $m(x) \geqslant \alpha'$ and $\alpha \geqslant M(x)$, which imply $m(x) > M(x)$. This contradicts property (1). $\square$

This claim together with the Nachbin theorem [1, p. 36] prove the following version of Nachbin's extension theorem:

Theorem 1. *Let X be a normally preordered space and let $D \subset X$ be a closed set. Let $f : D \to R$ be a continuous and monotonic function. Then f has an extension if* *and only if property (1) is satisfied.*

Remark. Nachbin's extension theorem assumes the boundedness of function f. Howvere, the following simple observation removes this assumption:
For an arbitrary function $f : D \to R$ and for every increasing homeomorphism $\phi : R \to (0, 1)$

$$\phi \circ m_f = m_{\phi \circ f} \text{ and } \phi \circ M_f = M_{\phi \circ f}.$$

Classical examples of normally partially ordered spaces are Euclidean spaces ordered with the componentwise order. Recall that a preorder is called a*partial order* if it is antisymmetric. For two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in R^n we write $x \leqslant y$ if $x_i \leqslant y_i$ for all $i = 1, \ldots, n$; $x \leq y$ if $x \leqslant y$ and $x \neq y$, and $x < y$ if $x_i < y_i$ for all $i = 1, \ldots, n$. We also write $x \geqslant y,\ x \geq y,\ x > y$ if $y \leqslant x,\ y \leq x,\ y < x$, respectively. Since the preorder $\geqslant$ on R^n is antisymmetric preorder, $(R^n, \geqslant)$ is a normally partially ordered space. Denote by e the vector in R^n all of whose components are 1, and $e_k\, (k = 1, \ldots, n)$ the vector in R^n whose k-th component is 1 and all other components are 0.

3. Extension of increasing functions

In this section we consider the problem of extending a continuous, (weakly) increasing function $f : D \to R$, where D is a closed set in R^n, into the entire space R^n. Here we give a necessary and sufficient condition for the existence of such extensions. As the equivalence of the Nachbin property and property (1) is proved above, the following extension result is a modification of the Nachbin theorem [1, p. 36] for the case of Euclidean spaces.

Theorem 2. *Let $D \subset R^n$ be a nonempty, closed set and $f : D \to R$ a continuous, increasing function. Then there exists a continuous, increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for $x \in D$ if and only if function f satisfies inequality (1).*

Proof: If there exists a continuous, increasing extension F of function f, then obviously $m(x) \leqslant F(x + re)$ and $M(x) \geqslant F(x - re)$ for all $x \in R^n$ and all $r > 0$. Since, F is continuous it follows that $m(x) \leqslant F(x) \leqslant M(x)$ and hence $m(x) \leqslant M(x)$ for all $x \in R^n$.

Conversely, we now prove now that if property (1) holds, then there exists an extension F as stated in the theorem.

Claim: *$m(\cdot)$ is upper semicontinuous and $M(\cdot)$ is lower semicontinuous. Hence, the correspondence $x \mapsto [m(x), M(x)]$, $x \in R^2$ is lower hemicontinuous.*

Proof: Fix $x_0 \in R^n$. Let $\varepsilon > 0$. By the definition of $m(\cdot)$ there exists a positive number r such that

$$\sup\{f(z) : z \in D, z \leqslant x_0 + 2re\} < m(x_0) + \varepsilon.$$

Since for each $x \in B_r(x_0)$ the inequality $z \leqslant x + re$ implies $z \leqslant x_0 + 2re$, we have $m(x) < m(x_0) + \varepsilon$ for each $x \in B_r(x_0)$. That is, $m(\cdot)$ is upper semicontinuous. The lower semicontinuity of $M(\cdot)$ is proved similarly.

We extend first f into $D \cup K_1$, where $K_1 = [-1, 1]^n$. By the Michael's selection theorem there exists a continuous function $g' : K_1 \to R$ such that $m(x) \leqslant g'(x) \leqslant M(x)$ for all $x \in K_1$. Since $m(x) = M(x) = f(x)$ for $x \in D \cap K_1$ we have $g'(x) = f(x)$ for $x \in D \cap K_1$. Set

$$g(x) = \max\{g'(z) : z \in K_1,\ z \leqslant x\} \ \ for \ \ x \in K_1.$$

It is an easy matter to show that g is continuous and increasing. Obviously $m(x) \leq g(x)$ for $x \in K_1$. Moreover, since $M(\cdot)$ is increasing we have $g(x) \leq M(x)$ for $x \in K_1$. Thus

$$m(x) \leqslant g(x) \leqslant M(x) \ \ for \ \ x \in K_1.$$

Indeed, we claim that the function $f_1 : D \cup K_1 \to R$ defined as $f(x)$ for $x \in D \setminus K_1$, and as $g(x)$ for $x \in K_1$ is continuous and increasing. Obviously, f_1 is continuous on $D \setminus K_1$. Let $x_0 \in K_1$ and $\{x_k\}$ be a sequence in $D \setminus K_1$ converging to x_0. Since D is assumed to be closed, $x_0 \in D$. Therefore $g'(x_0) = f(x_0)$. By the definition of function g, $g(x_0) = f(x_0)$. Since f is continuous on D it follows that $f_1(x_k) = f(x_k) \to f(x_0) = g(x_0) = f_1(x_0)$. Now show that f_1 is increasing. Take $x \in K_1$ and $y \in D$, $y \leqslant x$. Then, $f_1(y) = f(y) \leqslant m(x) \leqslant f_1(x)$. Take $x \in K_1$ and $y \in D$, $y \geqslant x$. Then, $f_1(y) = M(y) \geqslant M(x) \geqslant f_1(x)$. Since, $f_1|_{K_1}$ and $f_1|_D$ are increasing it follows that f_1 is increasing. So we constructed a continuous and increasing extension f_1 of function f into $D_1 = D \cup K_1$.

Now we show that f_1 has the property

$$m_{f_1}(x) \leqslant M_{f_1}(x) \text{ for all } x \in R^n. \tag{2}$$

For any point x in R^n denote by $\hat{x}$ the point in K_1 that is closest to x. We consider four cases:

Case 1: $x \in R^n \setminus [(K_1 + R^n_+) \cup (K_1 - R^n_+)]$. Clearly, $m_{f_1}(x) = m_f(x)$ and $M_{f_1}(x) = M_f(x)$, and hence $m_{f_1}(x) \leqslant M_{f_1}(x)$.

Case 2: $x \in (K_1 + R^n_+) \setminus K_1$. By the monotonicity and continuity of function f_1 we have $m_{f_1}(x) = \max\{f_1(\hat{x}), m_f(x)\}$. This and the inequalities $f_1(\hat{x}) \leqslant M_f(\hat{x}) \leqslant M_f(x)$ and $m_f(x) \leqslant M_f(x)$ and the equality $M_f(x) = M_{f_1}(x)$ imply $m_{f_1}(x) \leqslant M_{f_1}(x)$.

Case 3: $x \in (K_1 - R^n_+) \setminus K_1$. Again by th monotonicity and continuity of function f^1 we have $M_f(x) = \min\{f_1(\hat{x}), M_f(x)\}$. This together with the inequalities $f_1(\hat{x}) \geqslant m_f(\hat{x}) \geqslant m_f(x)$ and $M_f(x) \geqslant m_f(x)$, and the equality $m_f(x) = m_{f_1}(x)$ imply $m_{f_1}(x) \leqslant M_{f_1}(x)$.

Case 4: $x \in K_1$. We have $M_f(x) \leqslant f_1(x) \leqslant m_f(x)$ and hence $M_f(x) = f_1(x) = m_f(x)$. Obviously, $M_{f_1}(x) \geqslant M_f(x)$ and $m_{f_1}(x) \leqslant m_f(x)$. Therefore, $m_{f_1}(x) \leqslant M_{f_1}(x)$.

Since, $f_1 : D_1 \to R$, where $D_1 \subset R^n$ is a closed set, is continuous and possesses property (2), by the above argument we can extend f_1 into $D_2 = D \cup K_2$, where $K_2 = [-2, 2]^n$. Proceeding in this manner we will obtain a continuous and increasing extension F of function f into the whole space R^n. □

Corollary 1. *Let $D \subset R^n$ be a nonempty, compact set and $f : D \to R$ a continuous, increasing function. Then there exists a continuous, increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for $x \in D$.*

Proof: It is easy to see that when D is nonempty and compact, functions M and m can be defined as

$$m(x) = \max\{f(z) : z \in D \cap L(x)\} \text{ and } M(x) = \min\{f(z) : z \in D \cap U(x)\} \tag{3}$$

for $x \in R^n$, where $L(x) = x + R^n_-$ and $U(x) = x + R^n_+$.
By the monotonicity of f, obviously $m(x) \leqslant M(x)$ for all $x \in R^n$. Theorem 2 applies. □

4. Extension of strictly increasing functions

We shall consider R^n with the square-norm $||x|| = \max\{|x_i|,\ i = 1, \dots, n\}$. For a nonempty set $E \subset R^n$ and a point $x \in R^n$ the distance between them is defined as dist $(x, E) = \inf\{||x - y|| : y \in E\}$. For a set $E \subset R^n$, $\mathring{E}$ and ∂E will denote its interior and its boundary, respectively.

Throughout this section, K, possibly equipped with indexes, will denote a cube in R^n with the edges parallel to coordinate axes. A face of the cube $K = [a_i, b_i]^n$ in R^n is called a *lower (upper)* face if it contains the smallest (greatest) vertex $a = (a_1, \dots, a_n)$ ($b = (b_1, \dots, b_n)$). The word 'extension' will mean 'continuous strictly increasing extension'.
Further for $t \in R$ the interval (t, t) will mean the singleton $\{t\}$.

Proof of the following statement is straightforward.

Claim 1. *The supremum and infimum of a family of equicontinuous functions defined on a set $E \subset R^n$ is continuous.*

Claim 2. *Let $K \subset R^n$ be a cube and $F_1, F_2 : K \to R$ continuous, monotone functions such that $F_1(x) < F_2(x)$ for all $x \in K$, and $f : \partial K \to R$ continuous, strictly increasing function, such that*

$$f(x) \in (F_1(x), F_2(x)) \ \text{ for } x \in \partial K.$$

Then there exists a continuous, strictly increasing extension F of function f into K such that

$$F(x) \in (F_1(x), F_2(x)) \text{ for all } x \in K.$$

Proof: Define $\bar{m}$, $\bar{M} : K \to R$ as

$$\bar{m}(x) = \max\{f(z) : z \in \partial K,\ z \leqslant x\} \ \text{ and } \bar{M}(x) = \min\{f(z) : z \in \partial K,\ z \geqslant x\}.$$

Note that functions $\bar{m}$, and $\bar{M}$ are monotone and

$$\bar{m}|_{\partial K} = \bar{M}|_{\partial K} = f.$$

Moreover, $\bar{m}$ is upper semicontinuous and $\bar{M}$ is lower semicontinuous, $\bar{m}$ is continuous on $(\underline{\partial} K) \cup \overset{\circ}{K}$ and $\bar{M}$ continuous on $(\overline{\partial} K) \cup \overset{\circ}{K}$, where $\underline{\partial} K$ is the union of the lower faces of K and $\overline{\partial} K$ is the union of the upper faces of K. Set

$$m'(x) = \max\{\bar{m}(x), F_1(x)\}, \text{ and } M'(x) = \min\{\bar{M}(x), F_2(x)\}.$$

Functions m', and M' are monotone,

$$F_1(x) \leqslant m'(x) < F_2(x),\ F_1(x) < M'(x) \leqslant F_2(x) \text{ for all } x \in K. \tag{4}$$

Moreover,

$$m'(x) < M'(x) \text{ for } x \in \overset{\circ}{K} \text{ and } m'(x) = M'(x) = f(x) \text{ for } x \in \partial K. \tag{5}$$

Set

$$F(x) = \Lambda(x)M'(x) + (1-\Lambda(x))m'(x) \text{ for } x \in K, \tag{6}$$

where $\Lambda : K \to R$ is defined as

$$\Lambda(x) = \begin{cases} 1 & \text{for} \qquad x \in \overline{\partial}K, \\ \frac{\text{dist}\,(x,\underline{\partial}K)}{\text{dist}\,(x,\underline{\partial}K)+\text{dist}\,(x,\overline{\partial}K)} & \text{otherwise.} \end{cases}$$

It follows that F is continuous, $F|_{\partial K} = f$. Since Λ is strictly increasing on $\mathring{K}$, and functions m', and M' are monotone it follows that $F(x)$ is strictly increasing on $\mathring{K}$. This, continuity of F and strict monotonicity of $F|_{\partial K}$ imply that F is strictly increasing.
From the relations (4) and (6), $F(x) \in (F_1(x), F_2(x))$ for all $x \in \mathring{K}$. From the relations (5) and (6), we have $F(x) = f(x) \in (F_1(x), F_2(x))$ for $x \in \partial K$. Thus $F(x) \in (F_1(x), F_2(x))$ for all $x \in K$. □

Claim 3: *Let K be a cube in R^n and $G_1, G_2 : K \to R$ increasing functions such that*

$$G_1(x) < G_2(x) \textit{ for all } x \in K.$$

Moreover, let G_1 be upper semicontinuous, G_2 lower semicontinuous, and $f : C \to R$, where C is a closed subset (possibly empty) of K, be a continuous function such that $G_1(x) < f(x) < G_2(x)$ for every x in C. Then there exist continuous increasing functions $F_1, F_2 : K \to R$ such that

$$G_1(x) \leqslant F_1(x) < F_2(x) \leqslant G_2(x) \textit{ for all } x \in K \tag{7}$$

and

$$F_1(x) < f(x) < F_2(x) \textit{ for all } x \in C. \tag{8}$$

Proof: Define

$$G_1'(x) = \begin{cases} f(x) & \text{for} \quad x \in C, \\ G_1(x) & \text{for} \quad x \in K \setminus C. \end{cases}$$

Function G_1' is upper semicontinuous and $G_1'(x) < G_2(x)$ for all $x \in K$. Then, $d = \min\{G_2(x) - G_1'(x) : x \in K\} > 0$. By Michael's selection theorem there exists a continuous function F_2' on K such that

$$G_1'(x) + d \leqslant F_2'(x) \leqslant G_2(x) \text{ for all } x \in K.$$

Define

$$F_2(x) = \max\{F_2'(z) : z \in K, z \leqslant x\} \text{ for } x \in K.$$

Function F_2 is continuous, increasing and

$$G_1'(x) + d \leqslant F_2(x) \leqslant G_2(x) \text{ for all } x \in K. \tag{9}$$

The second inequality in (9) follows from the monotonicity of G_2. In particular,

$$f(x) < F_2(x) \text{ for all } x \in C. \tag{10}$$

Define

$$F_2''(x) = \begin{cases} f(x) & \text{for} \quad x \in C, \\ F_2(x) & \text{for} \quad x \in K \setminus C. \end{cases}$$

Function F_2'' is lower semicontinuous and $F_2''(x) > G_1(x)$ for $x \in K$. Then, $d' = \min\{F_2''(x) - G_1(x) : x \in K\} > 0$. By Michael's selection theorem there exists a continuous function F_1' on K such that

$$G_1(x) \leqslant F_1'(x) \leqslant F_2''(x) - d' \text{ for all } x \in K.$$

Set

$$F_1(x) = \min\{F_1'(z) : z \in K,\ z \geqslant x\} \text{ for all } x \in K.$$

Function F_1 is continuous, increasing and

$$G_1(x) \leqslant F_1(x) \leqslant F_2(x) - d' \text{ for all } x \in K. \tag{11}$$

The first inequality in (11) follows from the monotonicity of G_1. The second inequality implies $F_1(x) < F_2(x)$ for all $x \in K$. Also

$$F_1(x) < f(x) - d' \text{ for all } x \in C. \tag{12}$$

Now inequality (7) follows from inequalities (9) and (11), and inequality (8) from inequalities (10) and (12). □

Basic Lemma. *Let $K = \Pi^n[a_i, b_i]^n$ be a cube in R^n, $C \subset K$ the union of a family (possibly empty) of faces of K, and $f : C \to R$ a continuous, strictly increasing function. Let $F_1, F_2 : K \to R$ be continuous, increasing functions such that $F_1(x) < F_2(x)$ for all $x \in K$ and $F_1(x) < f(x) < F_2(x)$ for all $x \in C$. Then, there exists a continuous, strictly increasing extension F of function f into K such that*

$$F(x) \in (F_1(x), F_2(x)) \text{ for all } x \in K.$$

Proof. Assume without loss of generality that $K = [0, 1]^n$. Arrange all the faces of K into a sequence $K^1, K^2, \ldots, K^s$ so that each face comes before all faces of larger dimensions and faces of the same dimension are arranged arbitrarily with respect to each other and $K^1 = \{a\}$. If $a \notin C$, then we set $f(a)$ to be any number in the interval $(F_1'(a), F_2'(a))$, where $F_1'(a) = F_1(a)$ if there exists no $z \in C$ such that $a \geqslant z$, and

$$F_1'(a) = \max[\{f(y) : y \in C,\ y \leqslant a\} \cup \{F_1(a)\}] \text{ otherwise,}$$

and $F_2'(a) = F_2(a)$ if there exists no $z \in C$ such that $z \geqslant a$, and

$$F_2'(a) = \min[\{f(y) : y \in C,\ y \geqslant a\} \cup \{F_2(a)\}] \text{ otherwise.}$$

Now assume function f is extended into all faces K^j for $j < i$. We denote this extension as f. For faces K', $K'' \in \mathcal{F}$ of the same dimension we say K' is *below* K'' if there exists a nonnegative vector x such that $K' + x = K''$ and denote this as $K' \prec K''$. In this case we also say K'' is *above* K' and write $K'' \succ K'$. Denote by $\mathcal{F}$ the set of all faces of K; $K_i = C \cup (\cup_{1 \leqslant j < i} K^j),\ i = 1, \ldots, s$, and $\mathcal{F}_b(K^i) = \{K' \in \mathcal{F} : K' \subset K_i \text{ and } K' \prec K^i\}$ and $\mathcal{F}_a(K^i) = \{K' \in \mathcal{F} : K' \subset K_i \text{ and } K' \succ K^i\}$. For each $K' \in \mathcal{F}_b(K^i)$ ($K' \in \mathcal{F}_a(K^i)$) we denote by $e(K')$ the nonnegative vector such that $K^i - e(K') = K'$ ($K^i + e(K') = K'$). Define functions $F_1', F_2' : K^i \to R$ as

$$F_1'(x) = \max\{f(x - e(K')) : K' \in \mathcal{F}_b(K^i)\} \cup \{F_1(x)\}]$$

and

$$F_2'(x) = \min[\{f(x + e(K')) : K' \in \mathcal{F}_a(K^i)\} \cup \{F_2(x)\}],$$

Functions F_1' and F_2' are continuous, increasing and $F_1'(x) < F_2'(x)$ for all $x \in K^i$, and $F_1'(x) < f(x) < F_2'(x)$ for all $x \in \partial K^i$. By Claim 2 there exists an extension of function $f|_{\partial K^i}$ into K^i. So we have extended function f into $K_{i+1} = K_i \cup K^i$. This extension is continuous and strictly increasing.

The above inductive procedure extends function f into $K^s = K$. □

Theorem 3. *Let $D \subset R^n$ be a nonempty, closed set and $f : D \to R$ a continuous, strictly increasing function. Then there exists a continuous, strictly increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for $x \in D$ if and only if function f satisfies the following condition:*

$$m(x) \leqslant M(x) \text{ for all } x \in D \text{ and } m(x) < M(x) \text{ for all } x \notin D. \tag{13}$$

Proof: First prove that condition (13) is necessary for the existence of an extension as in the theorem. Let $F : R^n \to R$ be an extension of f as in the theorem, and $x_0 \in D^c$. Then, $[x_0 - r, x_0 + r]^n \subset D^c$ for some $r > 0$. Now by the strict monotonicity $F(x_0) > F(x_0 - re_k),\ k = 1, \ldots, n$. By the continuity of F there exists $\delta_1 \in (0, r)$ such that

$$F(x_0) > F(x_0 - re_k + \delta_1 e),\ k = 1, \ldots, n.$$

Since $D \cap \{x \in R^n : x \leqslant x_0 + \delta_1 e\} \subset \cup_{k=1}^n \{x \in R^n : x \leqslant x_0 - re_k + \delta_1 e\}$ and F is monotonic and $F|_D = f$, it follows that

$$\sup\{f(z) : z \in D,\ z \leqslant x_0 + \delta_1 e\} < F(x_0). \tag{14}$$

In a similar way it is shown that

$$F(x_0) > \inf\{f(z) : z \in D,\ z \geqslant x_0 - \delta_2 e\} \tag{15}$$

for some $\delta_2 > 0$.

It follows from the equations (14) and (15) that $m(x_0) < M(x_0)$.

Now, show that if condition (13) is satisfied then there exists an extension of function f into R^n. By Theorem 2 there exists a continuous, increasing extension G of function f into R^n. Denote $K = [-1, 1]^n$ and set

$$\bar{\varphi}(\delta) = \max\{|G(x) - G(y)| : x, y \in K, ||x - y|| \leqslant \delta\}$$

for $\delta \geqslant 0$. Clearly $\bar{\varphi} : R_+ \to R$ is an increasing, continuous function and $\bar{\varphi}(0) = 0$. Let $\varphi : R_+ \to R$ be a strictly increasing, continuous function such that $\varphi(\delta) \geqslant \bar{\varphi}(\delta)$ for all $\delta \geqslant 0$ and $\varphi(0) = 0$. For $y \in D \cap K$ define a function $\psi_y : K \to R$ as

$$\psi_y(x) = \begin{cases} f(y) & for \quad x \geqslant y, \\ f(y) - 2\varphi(dist\,(x, y + R^n_+)) & for \quad x \ngeqslant y. \end{cases}$$

Obviously, ψ_y is a continuous, increasing function for each $y \in D \cap K$. Set
$F_1(x) = \sup\{\psi_y(x) : y \in D\}$ for $x \in K$.

Similarly, define functions $\chi_y : K \to R$ as

$$\chi_y(x) = \begin{cases} f(y) & for \quad x \leqslant y, \\ f(y) + 2\varphi(\mathrm{dist}\,(x, y + R^n_-)) & for \quad x \nleqslant y. \end{cases}$$

Set $F_2(x) = \inf\{\chi_y(x) : y \in D \cap K\}$ for $x \in K$. Functions F_1 and F_2 are obviously increasing, and are continuous by Claim 1. Moreover,

$$F_1(x) = F_2(x) = f(x) \quad for \quad x \in D \cap K, \tag{16}$$

and

$$F_1(x) < G(x) < F_2(x) \quad \text{for} \quad x \in K \setminus D. \tag{17}$$

Define functions $F_1'(x) = \max\{F_1(x), m(x)\}$ and $F_2'(x) = \min\{F_2(x), M(x)\}$ for $x \in K$, and define a correspondence $\mathcal{F} : K \Rightarrow R$ as

$$\mathcal{F}(x) = [F_1'(x), F_2'(x)] \quad \text{for} \quad x \in K.$$

Conditions (13) imply that $m(x) = M(x) = f(x)$ for all $x \in D$. From this and equation (16),

$$F_1'(x) = F_2'(x) = f(x) \quad \text{for} \quad x \in D \cap K. \tag{18}$$

Thus, if we show that correspondence $\mathcal{F}$ has a strictly increasing continuous selection, then we are done.

For each $k \in N$ consider the family of all hyperplanes parallel to the coordinate hyperplanes defined by equations $x_i = \frac{m}{2^k},\ m \in Z,\ i = 1, \ldots, n$. For each $k \in N$ these hyperplanes divide R^n into cubes of the side $\frac{1}{2^k}$. We denote this collection of cubes by O_k. We call number k the *rank* of the cubes in O_k. The set $K \setminus D$ is divided into cubes in $O = \cup_{k \in N} O_k$ with the disjoint interiors in the following way. First we pick all cubes in O_1 that are contained in K. Then we pick all cubes in O_2 that are contained in K and are not contained in the cubes in O_1 that were picked in the first step, and so on. Since there are a countable number of cubes chosen we can arrange them into a sequence $K_1, K_2, \ldots$, so that $rank\ K_j \leqslant rank\ K_{j+1}$ for each $j \in N$. By the relations (17) and (18) functions F_1' and F_2' satisfy the assumptions of Claim 3. By this claim there exist continuous increasing functions H_1 and H_2 on K_1 such that

$$F_1'(x) \leqslant H_1(x) < H_2(x) \leqslant F_2'(x) \text{ for all } x \in K_1.$$

Now by the Lemma there exists a continuous, strictly increasing function f_1 on K_1 such that

$$H_1(x) < f_1(x) < H_2(x) \ \text{ for all } \ x \in K_1.$$

Next, by way of induction assume f_l to be a continuous, strictly increasing extension of f into $D_l = D \cup (\cup_{j=1}^l K_j)$ for $l \in N$, satisfying $f_l(x) \in (F_1(x), F_2(x))$ for all $x \in D_l$. By the Lemma there exists an extension of function f_l into K_{l+1}.

The above inductive procedure extends function f into the union $K \cup D$. This extension, denoted as f^1, is continuous on $K^1 \setminus D$, because the family of cubes $\{K_1, K_2, \ldots\}$ is locally finite and f_1 is continuous on each of these cubes. Continuity of f^1 on $K \cup D$ follows from the continuity of F_1 and F_2, the inequality $F_1(x) \leqslant f^1(x) \leqslant F_2(x)$ for all $x \in K$, and the equations (16). Finally, f^1 is strictly increasing because it is strictly increasing on each of the sets D_l $(l = 1, 2 \ldots)$, $D_l \subset D_{l+1}$ $(l = 1, 2, \ldots)$, and its domain is $D^1 = \cup_l D_l$.

Now we assert that for the extension f^1

$$m_{f^1}(x) = M_{f^1}(x) \ \text{ for } \ x \in D^1 \tag{19}$$

and

$$m_{f^1}(x) < M_{f^1}(x) \text{ for } x \notin D^1. \tag{20}$$

The proof of (19) is simple and we omit it. To prove (20) consider three cases.

Case 1: $x \in (R^n \setminus D^1) \setminus [(K^1 + R^n_+) \cup (K^1 - R^n_+)]$. Clearly,

$m_{f^1}(x) = m_f(x)$ and $M_{f^1}(x) = M_f(x)$ and hence $m_{f^1}(x) < M_{f^1}(x)$.

For each point $x \in R^n$ denote by $\hat{x}$ the point in K closest to x. Obviously, the mapping $x \mapsto \hat{x}$ is continuous and increasing.

Case 2: $x \in (R^n \setminus D^1) \cap (K + R^n_+)$. By the monotonicity and continuity of function f^1 we have $m_{f^1}(x) = \max\{f^1(\hat{x}), m_f(x)\}$. If $\hat{x} \in D$, then $f^1(\hat{x}) = f(\hat{x})$ and hence $m_{f^1}(x) = m_f(x)$. Obviously, $M_{f^1}(x) = M_f(x)$ and by the assumption $m_f(x) < M_f(x)$. So $m_{f^1}(x) < M_{f^1}(x)$. If $\hat{x} \notin D$, then by the construction of function f^1, $f^1(\hat{x}) > m_f(x)$ and hence $m_{f^1}(\hat{x}) = f^1(\hat{x})$. Since $f^1(\hat{x}) < M_f(\hat{x}) \leqslant M_f(x) = M_{f^1}(x)$ it follows that $m_{f^1}(x) < M_{f^1}(x)$.

Case 3: $x \in (R^n \setminus D^1) \cap (K + R^n_-)$. By the monotonicity and continuity of function f^1 we have $M_{f^1}(x) = \min\{f^1(\hat{x}), M_f(x)\}$. If $\hat{x} \in D$, then $f^1(\hat{x}) = f(\hat{x})$ and hence $M_{f^1}(x) = M_f(x)$. Obviously, $m_{f^1}(x) = m_f(x)$ and by the assumption $m_f(x) < M_f(x)$. So $m_{f^1}(x) < M_{f^1}(x)$. If $\hat{x} \notin D$, then by the construction of function f^1 we have $f^1(\hat{x}) < M_f(x)$ and hence $M_{f^1}(x) = f^1(\hat{x})$. Since $f^1(\hat{x}) > m_f(\hat{x}) \geqslant m_f(x) = m_{f^1}(x)$, it follows that $m_f(x) < M_{f^1}(x)$.

Now by the above arguments there exists an extension f^2 of function f^1 into $[-2, 2]^n \cup D$ for which the conditions (13) are satisfied. Proceeding in this way we obtain a continuous, strictly increasing function $F : R^n \to R$ which is an extension of f. $\square$

Corollary 2. *Let $D \subset R^n$ be a nonempty, compact set, and $f : D \to R$ a continuous, strictly increasing function. Then there exists a continuous, strictly increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for $x \in D$.*

Proof: As was noted in the proof of Corollary 1, functions m and M can be equivalently defined by the formulas in (3). It is clear from the formulas in (3) that $m(x) = M(x)$ for $x \in D$ and $m(x) < M(x)$ for $x \notin D$. Theorem 3 applies. $\square$

The formulas (3) may hold for some unbounded closed domains as well. However this alone is not sufficient for assumption (13) of Theorem 3 to hold. If in addition the sets $D \cap L(x)$ and $D \cap U(x)$ have the compact sets of $\leqslant -$maximal and $\leqslant -$minimal elements, respectively, then assumption (13) holds. Examples of domains with this property are subsets of Z^n, where $Z = \{0, \pm 1, \pm 2, \ldots\}$.

Corollary 3. *Let $f : D \to R$, where $D \subset Z^n$, be a strictly increasing function. Then there exists a continuous, strictly increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for all $x \in D$. In particular, for every strictly increasing function $f : D \to R$ there exists a strictly increasing extension of function f into Z^n.*

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